

Stochastic Model of a One-Dimensional Fluid

Sezar Fesciyan¹ and H. L. Frisch¹

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A one-dimensional point process with correlations constructed via a geometrical rule is shown to behave like a fluid at equilibrium. The equation of state is calculated and the "inverse problem" of finding an interaction potential underlying the system is considered. The effective potential is found to be dependent on macroscopic parameters via a dependence on the density of the system.

KEY WORDS: Point process; particle correlations; effective potential.

1. INTRODUCTION

Given the interaction potential $\phi(r)$, the standard methods of statistical mechanics allow in principle the calculation of an equation of state via the partition function suitable for a given set of macroscopic parameters. This can actually be carried out exactly in one dimension. Suppose, however, that one does not know the forces, but that the statistical properties (such as particle correlations) of a system are somehow given. One question would be: What are the interactions (if any) that would give rise to the given statistical properties for a system at equilibrium? And another is: What is the equation of state? These questions fall under the so-called "inverse problem of statistical mechanics" that has occasionally been studied in the literature. For example, the problem of inverting the second virial coefficient to obtain the pair

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¹ Department of Chemistry, State University of New York at Albany, Albany, New York.

potential has been considered.⁽¹⁻³⁾ Kunin and Frisch⁽⁴⁾ have found pair potentials and equations of state that would be consistent with certain “physically reasonable” choices of candidate pair correlation functions. However, one does not in general know what specific class of functions could serve as pair correlation functions to describe systems at equilibrium. There are no useful criteria that are specific enough to help construct analytically a particular nontrivial function which is guaranteed to be a pair correlation function for some potential. In fact, given a point process (with some physically implied restrictions such as homogeneity, etc.) with correlations, it is not clear that it can always be identified with an ensemble describing an equilibrium system of particles with definite forces among them. It is only in the lattice case where it has been shown by Spitzer⁽⁵⁾ that every reasonable two-valued process (so-called Markovian random fields) which can construct on the lattice corresponds to a Gibbsian ensemble for some pair potential.

In this paper we use a method of geometrical construction to obtain a system with simple correlations and discuss to what extent it can represent a physical system. In Section 2 we obtain the equation of state, and in Section 3 obtain the underlying potential and discuss its peculiarities.

2. THE EQUATION OF STATE

A way to obtain the equation of state purely from the statistical properties of the point process defining the system while bypassing any specification of the interaction potential would be to calculate the probability $P_0(V)$ that a region of volume V will be found to be devoid of particles. Then for large enough volumes V , $P_0(V)$ can be identified with $e^{-\beta pV}$ ($\beta = 1/kT$), from which one can obtain the equation of state as

$$\beta p = - \lim_{V \rightarrow \infty} (1/V) \ln P_0(V) \quad (1)$$

The simplest example is when we have a Poisson distribution of points of density ρ , so that $P_0(V) = e^{-\rho V}$, from which follows the ideal gas law $\beta p = \rho$.

The system we study below has very simple correlations by construction. It has the advantage that $P_0(V)$ is easy to calculate. The idea behind the construction is a variation of what can be found in a paper by Gilbert,⁽⁶⁾ where another mathematical model of a fluid is constructed via a procedure similar to the one used below. We start with a Poisson process of points along the infinite one-dimensional line (an ideal gas) with average density ρ . To every configuration X of these uncorrelated points we associate a configuration \hat{X} of correlated points according to the following rule: A point $x \in X$ belongs

to \hat{X} if and only if there exist no other points $y \in X, y < x$, such that $x - y < \sigma$, where σ is some fixed length. In other words, given an equilibrium configuration of a one-dimensional ideal gas with average density ρ , we imagine a rod of length σ centered at each of its points. We keep only those rods that do not overlap with another rod to the left of them, and erase everything else. We are left with a configuration of rods with average density $\alpha = \rho e^{-\rho\sigma}$ and very simple correlations. Indeed, the joint probability $\rho^{(2)}(x_1, x_2) dx_1 dx_2$ of a rod remaining at x_1 and another at x_2 is the same as the probability that there are no Poisson points in the interval $(x_1 - \sigma, x_1)$ and that there is a Poisson point in the interval $(x_1, x_1 + dx_1)$, and similarly for $x_2 > x_1$. Thus

$$\rho^{(2)}(x_1, x_2) dx_1 dx_2 = \begin{cases} (e^{-\rho\sigma} \rho dx_1)(e^{-\rho\sigma} \rho dx_2) & \text{for } x_2 - x_1 > \sigma \\ 0 & \text{for } x_2 - x_1 \leq \sigma \end{cases}$$

and hence

$$g(r) = \begin{cases} 1, & r > \sigma \\ 0, & r \leq \sigma \end{cases} \tag{2}$$

Furthermore, $g^{(n)}(r_1, \dots, r_n) = \prod_{i=1}^{n-1} g(r_{i+1} - r_i)$ also holds. One feature is that the system thus constructed has a maximum density $\alpha_{\max} = 1/e\sigma$ which is less than the maximum density of a real, hard-rod system by a factor of $1/e$. In general this occurs with other stochastic models similarly constructed, as it does with Gilbert's⁽⁶⁾ model of a fluid. Widom⁽⁷⁾ has also studied a stochastic model with a maximum density less than the hard-rod, close-packed density. He singles out this characteristic to emphasize the fact that the system thus constructed cannot be identified with a real, hard-rod system at equilibrium. It seems plausible, however, that the set of all configurations $\{\hat{X}\}$ constructed, together with the probabilities $P(\hat{X})$ that are naturally attached to each, could correspond to the equilibrium ensemble $\{\hat{X}, P\}$ for *some* real system whose interaction potential remains to be determined. This is the point of view we pursue here. The interactions required to make $\{\hat{X}, P\}$ an equilibrium ensemble will be considered in Section 3.

Given the above interpretation of our model we can determine a "thermodynamic pressure" by Eq. (1). Calculation of $P_0(L)$ involves the consideration of all configurations in $\{\hat{X}\}$ consistent with having no particles in a region of size L , defined by one fixed particle centered at $x = 0$ and one centered at $x = L$. This can only be if all Poisson points falling inside this region are erased in accordance with our rule, i.e., each Poisson point falling in $[0, L - \sigma)$ must have at least one other point to its left and within distance σ . In particular the point at the origin must have at least one Poisson point to its right within distance σ . Now, the probability of a configuration $\{q_1, \dots, q_n\}$

of n ordered Poisson points within a region of length $L - \sigma$ is given by $\rho^n e^{-\rho(L-\sigma)} dq_1 \dots dq_n$, so that $P_0(L)$ is given by

$$P_0(L) = \sum_{n=0}^{\infty} e^{-\rho(L-\sigma)} \rho^n \int_0^{L-\sigma} dq_n \int_0^{q_n} dq_{n-1} \dots \int_0^{q_2} dq_1 \times \epsilon[\sigma - q_1] \epsilon[\sigma - q_2 + q_1] \dots \epsilon[\sigma - q_n + q_{n-1}]$$

where $\epsilon[x]$ is the step function

$$\epsilon[x] = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Laplace-transforming, we have

$$P_0(L) = e^{-\rho(L-\sigma)} \mathcal{L}^{-1} \left\{ \frac{1}{s} \sum_{n=0}^{\infty} \rho^n \left(\frac{1 - e^{-\sigma s}}{s} \right)^n \right\} (L - \sigma)$$

Letting for simplicity $(L - \sigma)/\sigma \equiv \nu$, an integer, we have

$$\begin{aligned} P_0(\nu) &= e^{-\rho\sigma\nu} \mathcal{L}^{-1} \left\{ \frac{1}{s - \rho(1 - e^{-\sigma s})} \right\} (\nu\sigma) \\ &= e^{-\rho\sigma\nu} \mathcal{L}^{-1} \left\{ \sum_{n=0}^{\infty} (-)^n \frac{\rho^n e^{-\sigma sn}}{(s - \rho)^{n+1}} \right\} (\nu\sigma) \\ &= e^{-\rho\sigma\nu} \sum_{n=0}^{\infty} (-)^n (\rho\sigma e^{-\rho\sigma})^n \frac{(\nu - n)^n}{n!} e^{\rho\sigma\nu} \epsilon[\nu - n] \end{aligned}$$

Therefore

$$P_0(\nu) = \sum_{n=0}^{\nu} (-)^n [(\alpha\sigma)^n / n!] (\nu - n)^n \tag{3}$$

The asymptotic behavior of $P_0(\nu)$ for large ν is determined by the largest pole of its Laplace transform

$$\mathcal{L}[P_0(\nu)](s) = 1/(s + \alpha e^{-s\sigma}) \tag{4}$$

since $P_0(\nu) = \sum_{n=1}^{\infty} R_n(\nu)$, where $R_n(\nu)$ is the residue of

$$e^{z\nu\sigma} / (z + \alpha e^{-z\sigma}) \tag{5}$$

Let $z = -\gamma$ be the largest root of the denominator in (5), i.e., $\gamma e^{-\gamma\sigma} - \alpha = 0$. For $\alpha\sigma < 1/e$, $0 < \gamma < 1/\sigma$ is a real root which is a simple pole with the residue $e^{-\gamma\nu\sigma} / (1 - \gamma\sigma)$. For $\alpha\sigma = 1/e$ we have a double root at $z = -1/\sigma$ with the residue $2(\nu + \frac{1}{3})e^{-\nu}$. Then for $\nu \rightarrow \infty$, Eq. (1) allows the identification $p(\alpha) = \gamma(\alpha)$; therefore the pressure satisfies

$$pe^{-p\sigma} = \alpha \tag{6}$$

up to and including the maximum density. (We have set $\beta = 1$.) The equation of state (6) can also be obtained by solving the integral equation

$$P_0(L) = \begin{cases} 0, & L \leq \sigma \\ e^{-\rho L} + \int_0^\sigma e^{-\rho t} P_0(L-t) \rho dt, & L > \sigma \end{cases}$$

via Laplace transforms.⁽⁸⁾ The lower branch of (6) gives the pressure as a function of density. That α is the density in the thermodynamic sense, i.e., $\alpha = \langle N \rangle / L$, is shown in Section 3. Since $\alpha = \rho e^{-\rho\sigma}$ by definition, we see that $p = \rho$ ($\rho \leq 1/\sigma$) and $P_0(L) \sim e^{-\rho L}$ for large enough L . So, in effect, the process of elimination of Poisson points leading to this model keeps the magnitude of the pressure constant while lowering the density. This can be interpreted as being due to the effective repulsive forces that must be introduced between the particles to account for the correlations.

We can also obtain easily the virial expansion

$$\sigma p(\alpha) = \sum_{l=1}^{\infty} (l^{l-1}/l!)(\alpha\sigma)^l = [\alpha\sigma + (\alpha\sigma)^2 + \frac{3}{2}(\alpha\sigma)^3 + \dots] \quad (7)$$

which differs from the hard rod case beginning with the second virial coefficient. The series (7) has the radius of convergence $1/e$ and it converges to 1 there. Note that the inverse compressibility

$$\alpha(\partial p/\partial \alpha) = (1/\sigma) \sum_{l=1}^{\infty} (l^l/l!)(\alpha\sigma)^l$$

also has the radius of convergence $1/e$, but it diverges there. This reflects the fact that for $\alpha\sigma = 1/e$ the system is jammed and can no longer be compressed. We can also derive the fugacity from $\partial(\ln z)/\partial p = 1/\alpha$:

$$\ln \frac{z}{z_0} = \int_{p_0}^p dp \frac{e^{p\sigma}}{p} = \ln\left(\frac{p}{p_0}\right) + (p - p_0)\sigma + \frac{p^2 - p_0^2}{2 \cdot 2!} \sigma^2 + \dots$$

from which it follows that

$$z = p \exp\{\sigma p + [(\sigma p)^2/2 \cdot 2!] + [(\sigma p)^3/3 \cdot 3!] + \dots\}$$

This compares with the expression

$$z = p e^{\sigma p}$$

for a real, hard-rod system.

Concerning Eq. (6), we want to make one important point. If we were to blindly substitute the exactly known $g(r)$ into the compressibility equation, we would obtain for an equation of state

$$p(\alpha) = -(1/\sigma) \ln(1 - \alpha\sigma)$$

which is different from (6). But, as is shown by Widom,⁽⁹⁾ this can only be the equation of state of a system of particles with additive excluded volume; and this our system clearly is not. The resolution lies in the fact that as general as the compressibility equation is, it nevertheless does not hold for our model, because, as it will be shown in Section 3, the underlying potential depends explicitly on α (and hence the activity). In this connection we should point out that some of the unphysical results obtained by Kunkin and Frisch⁽⁴⁾ on the basis of perfectly reasonable assumptions are due to their use of the compressibility equation in a situation where it does not apply.

3. THE POTENTIAL

We now come back to the first question we raised in the introduction. Namely, what are the interactions that account for the behavior of our system? We can approach this problem via a consideration of the probability of a given configuration of particles. We let $P^{(\omega)}(x_1, \dots, x_N) dx_1 \dots dx_N$ be the probability that exactly N particles will be found at $(x_1, x_1 + dx_1)$, $(x_2, x_2 + dx_2), \dots, (x_N, x_N + dx_N)$ within a region L defined by two fixed particles at the origin and at $x = L$. Clearly $P^{(\omega)}(x_1, \dots, x_N) = 0$ whenever any of the $|x_i - x_{i-1}| \leq \sigma$. Otherwise, we have

$$P^{(\omega)}(x_1, \dots, x_N) dx_1 \dots dx_N = P_0^{(\omega)}(x_1) e^{-\rho\sigma} dx_1 P_0^{(\omega)}(x_2 - x_1) e^{-\rho\sigma} dx_2 \dots \times P_0^{(\omega)}(x_N - x_{N-1}) e^{-\rho\sigma} dx_N P_0^{(\omega)}(L - x_N) \quad (8)$$

where $P_0^{(\omega)}(x)$ is as defined in Section 2. More explicitly

$$P_0^{(\omega)}(x) = \begin{cases} S^{(\omega)}(x - \sigma) = \sum_{j=0}^{[(x-\sigma)/\sigma]} \frac{(-)^j}{j!} (\alpha\sigma)^j \left(\frac{x - \sigma}{\sigma} - j\right)^j, & x > \sigma \\ 0, & x \leq \sigma \end{cases} \quad (9)$$

which is the function in (3), except that the upper index is now written in terms of the continuous variable x ($[x] =$ largest integer $\leq x$). Equation (8) is just the probability that the region $(0, x_1 - \sigma)$ is empty and that there is a Poisson point at $(x_1, x_1 + dx_1)$ which is kept as a particle (this has probability $e^{-\rho\sigma} \rho dx_1$), etc. The product form is due to the especially simple properties of the underlying Poisson process (independence of events in nonintersecting intervals).

Now, to make contact with statistical mechanics, we would like to make the following identification:

$$P^{(\omega)}(x_1, \dots, x_N) = z^N e^{-U(x_1, \dots, x_N)} / \Xi(z, L) \quad (10)$$

where $\Xi = e^{pL}$ is the grand partition function, so that we set

$$\frac{z^N e^{-U(x_1, \dots, x_N)}}{\Xi(z(\alpha), L)} = \begin{cases} \alpha^N P_0^{(\alpha)}(x_1) P_0^{(\alpha)}(x_2 - x_1) \dots P_0^{(\alpha)}(x_N - x_{N-1}) P_0^{(\alpha)}(L - x_N) \\ 0 & \text{if any } |x_i - x_{i-1}| \leq \sigma \end{cases} \quad (11)$$

It is understood that this identification holds asymptotically in the limit of large L . It is now evident that the interaction potential $U(x_1, \dots, x_N)$ is a sum of nearest-neighbor pair interactions $\varphi^{(\alpha)}(x)$, where $\varphi^{(\alpha)}(x)$ has a hard core of size σ , and, as the superscript indicates, it is an effective pair potential explicitly dependent on the macroscopic parameter α . [In fact it must also be proportional to β to explain why no temperature dependence appears in the right-hand side of (1).] To obtain $\varphi^{(\alpha)}(x)$ explicitly, set $N = 2$ in Eq. (11); since z and Ξ are known, φ is determined (i.e., $\phi(x) = -\log \left[\left(\frac{\alpha}{z} \right) e^{px} P_0(x) \right]$).

From (11) let us calculate the probability $P^{(\alpha)}(N, L)$ that a region of size L will contain exactly N particles. This is

$$P^{(\alpha)}(N, L) = \alpha^N \int_{N\sigma}^{L-\sigma} dx_N \int_{(N-1)\sigma}^{x_N-\sigma} dx_{N-1} \dots \int_{2\sigma}^{x_3-\sigma} dx_2 \int_{\sigma}^{x_2-\sigma} dx_1 \\ \times P_0^{(\alpha)}(x_1) P_0^{(\alpha)}(x_2 - x_1) \dots P_0^{(\alpha)}(x_N - x_{N-1}) P_0^{(\alpha)}(L - x_N) \quad (12)$$

With a change of variables $y_j = x_j - j\sigma$ we have

$$P^{(\alpha)}(N, L) = \alpha^N \int_0^{L-(N+1)\sigma} dy_N \int_0^{y_N} dy_{N-1} \dots \int_0^{y_3} dy_2 \int_0^{y_2} dy_1 \\ \times P_0^{(\alpha)}(y_1 + \sigma) P_0^{(\alpha)}(y_2 - y_1 + \sigma) \dots \\ P_0^{(\alpha)}(y_N - y_{N-1} + \sigma) P_0^{(\alpha)}(L - y_N + \sigma) \quad (13)$$

or from Eq. (2)

$$P^{(\alpha)}(N, \tilde{L}) = \alpha^N [S^{(\alpha)} * S^{(\alpha)} * \dots * S^{(\alpha)}](\tilde{L}) \quad (14)$$

where the brackets contain the $(N+1)$ -fold convolution of $S(x)$ and $\tilde{L} = L - (N+1)\sigma$.

Now,

$$\mathcal{L}[S^{(\alpha)}(y)](s) = \sum_{j=0}^{\infty} (-)^j \alpha^j \frac{e^{-js}}{s^{j+1}}$$

Then

$$\mathcal{L}[P^{(\alpha)}(N, \tilde{L})](s) = \alpha^N \frac{1}{s^{N+1}} \left(\sum_{j=0}^{\infty} (-)^j \alpha^j \left(\frac{e^{-\sigma s}}{s} \right)^j \right)^{N+1} \\ = \alpha^N \frac{1}{s^{N+1}} \sum_{k=0}^{\infty} (-)^k \binom{N+k}{k} \alpha^k \left(\frac{e^{-\sigma s}}{s} \right)^k \quad (15)$$

and

$$P^{(\alpha)}(N, \tilde{L}) = \alpha^N \sum_{k=0}^{\infty} (-)^k \binom{N+k}{k} \alpha^k \frac{(\tilde{L} - \sigma k)^{N+k}}{(N+k)!} \epsilon[\tilde{L} - \sigma k] \quad (16)$$

or

$$P^{(\alpha)}(N, L) = \alpha^N \sum_{k=0}^{\lfloor (L - (N+1)\sigma)/\sigma \rfloor} (-)^k \alpha^k \frac{[L - (N+k+1)\sigma]^{N+k}}{N! k!} \quad (17)$$

We observe that

$$\begin{aligned} \sum_{N=0}^{\lfloor (L-\sigma)/\sigma \rfloor} P(N) &= \sum_{N=0}^{\lfloor (L-\sigma)/\sigma \rfloor} \alpha^N \sum_{k=0}^{\lfloor (L-\sigma)/\sigma - N \rfloor} (-)^k \alpha^k \frac{[L - (N+k+1)\sigma]^{N+k}}{N! k!} \\ &= \sum_{N=0}^{\lfloor (L-\sigma)/\sigma \rfloor} \sum_{M=N}^{\lfloor (L-\sigma)/\sigma \rfloor} (-)^{M-N} \alpha^M \frac{[L - (N+k+1)\sigma]^M}{N! (M-N)!} \\ &= \sum_{M=0}^{\lfloor (L-\sigma)/\sigma \rfloor} (-)^M \alpha^M \frac{(L - \sigma - M\sigma)^M}{M!} \sum_{N=0}^M (-)^N \binom{M}{N} \end{aligned}$$

but $\sum_{N=0}^M (-)^N \binom{M}{N} = 0$ for all $M \geq 1$, implying that $\sum_{N=0}^{\infty} P(N) = 1$, as it should. We can also see that

$$\begin{aligned} \langle N \rangle &= \sum_{N=0}^{\lfloor (L-\sigma)/\sigma \rfloor} NP(N) \\ &= \sum_{M=0}^{\lfloor (L-\sigma)/\sigma \rfloor} (-)^M \alpha^M \frac{[L - \sigma - M\sigma]^M}{M!} \sum_{N=0}^M (-)^N N \binom{M}{N} \\ &= \alpha(L - 2\sigma) \end{aligned}$$

since $\sum_{N=0}^M (-)^N N \binom{M}{N} = 0$ for all $M \geq 2$. Therefore for large enough L , we have

$$\alpha = \langle N \rangle / L$$

the thermodynamic density, as expected.

Our task is now to show that the interaction potential depends on the activity z , thus explaining why the usual compressibility equation would not yield the correct equation of state. Since z is a function of α , it is enough to show that the potential depends on α . Indeed, with $\alpha(p)$ and $z(p)$ obtained in Section 2, an application of Lagrange's theorem yields $\alpha(z) = \sum_{j=1}^{\infty} c_j z^j$ with

$$c_j = \frac{1}{j} \sum_{\substack{\sum i_k = M \\ \sum k i_k = j-1}} (-)^M \frac{(j+2)^{i_1}}{i_1!} \prod_{k=2}^l \frac{1}{i_k!} \left(\frac{j+k!}{k \cdot k!} \right)^{i_k} \quad (18)$$

To check for any α dependence in $U(x_1, \dots, x_N)$ of (10), we consider the expression

$$\Xi(z, L) P(N, L) / z^N = Z(N, L) / N!$$

Since ordinarily the right-hand side cannot depend on z ,

$$\frac{\partial}{\partial z} \left(\frac{\Xi(z, L)P(N, L)}{z^N} \right) = 0$$

holds identically. Below we show that this derivative does not vanish identically in our case. With $(\partial z/\partial \alpha) \neq 0$, it is enough to show

$$\frac{\partial}{\partial \alpha} \left(\frac{e^{p(\alpha)L}P^{(\omega)}(N, L)}{(z(\alpha))^N} \right) \neq 0$$

identically, where we have substituted $\Xi(z(\alpha), L) = 1/P_0(L) = e^{p(\alpha)L}$ as found in Section 2. In the differentiation we make use of the relation

$$\alpha \frac{\partial P^{(\omega)}(N, L)}{\partial \alpha} = NP^{(\omega)}(N, L) - (N + 1)P^{(\omega)}(N + 1, L) \quad (19)$$

which can be easily derived from (17). We have

$$\begin{aligned} & \frac{\partial}{\partial \alpha} [(z(\alpha))^{-N} e^{p(\alpha)L} P^{(\omega)}(N, L)] \\ &= -\frac{N}{z^{N+1}} \left(\frac{\partial z}{\partial \alpha} \right) e^{p(\alpha)L} P^{(\omega)}(N, L) + \frac{L}{z^N} \left(\frac{\partial p}{\partial \alpha} \right) e^{p(\alpha)L} P^{(\omega)}(N, L) \\ & \quad + \frac{e^{p(\alpha)L}}{z^N \alpha} [NP^{(\omega)}(N, L) - (N + 1)P^{(\omega)}(N + 1, L)] \end{aligned} \quad (20)$$

But $\partial z/\partial \alpha = (z/\alpha) \partial p/\partial \alpha$. Then Eq. (20) becomes

$$\left[L \left(\frac{\partial p}{\partial \alpha} \right) + N \frac{1 - (\partial p/\partial \alpha)}{\alpha} \right] P(N, L) - \frac{N + 1}{\alpha} P(N + 1, L)$$

From the behavior of $\partial p/\partial \alpha$ as found in Section 2 we see that there are N and L that make the first bracket negative for α such that

$$\alpha < e^{-p(\alpha)\sigma}/\sigma < N/L < 1/\sigma$$

Also, the second term is necessarily negative by virtue of the meaning of $P(N + 1, L)$. Therefore we have shown that $(\partial/\partial z)[\Xi P^{(\omega)}(N, L)/z^N]$ is not identically zero.

We have found that the forces among the particles in this model cannot really be described in purely microscopic terms. The effective potential does not seem to be the net result of two-body, three-body, etc., forces depending only on the particle coordinates. Despite this, in the infinite volume limit the system behaves like a perfectly reasonable thermodynamic system. In conclusion, we would like to suggest that the class of systems of interest to statistical mechanics is perhaps larger than those that are specified via any set of interaction potentials. We are presently engaged in studying other

stochastic models both in one and two dimensions, and are hopeful of finding interesting macroscopic behavior of physical significance (e.g., phase transitions). For the particular model studied above, our conjecture is that any direct two- or three-dimensional analog of it would also lead to a smooth pressure versus density curve defined up to a maximum density, i.e., qualitatively the same as in one dimension.

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